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Collapsing and monopole classes of 3-manifolds

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Abstract

By applying the L^2-estimate of the scalar curvature of a Riemannian 3-manifold with a Seiberg–Witten monopole class to a collapsing sequence of metrics, we obtain conditions to be a monopole class on certain 3-manifolds. This also gives a relation between a maximizing sequence of the Yamabe constants and the collapsing on a 3-manifold with a non-torsion monopole class.

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1. Introduction and statement of the main results

While the Seiberg–Witten theory gives information about the differential topology of the manifold independent of the Riemannian metric, it also tells us about the possible Riemannian geometry of the manifold. One of the well-known facts of this kind is that if a 3- or 4-manifold has a nontrivial Seiberg–Witten invariant, it cannot admit a metric of positive scalar curvature. This is an immediate consequence of the Weitzenböck formula. LeBrun exploited the Weitzenböck formula to produce various forms of the curvature estimate.

Theorem 1.1 (*LeBrun [11–13]*). *Let (X, g) be a smooth closed oriented Riemannian 4-manifold. Suppose that (X, g) has a solution of the Seiberg–Witten equations for a* Spin^c *structure with the first Chern class c_1. Then*

$$\int_X (s_-)^2 \mathrm{d}\mu \geq 32\pi^2 (c_1^+)^2,$$

where s_- denotes the pointwise minimum of zero and the scalar curvature s of g, and c_1^+ is the g-self-dual harmonic part of c_1. If c_1 is a monopole class, then

$$\int_X |r|^2 \mathrm{d}\mu \geq 16\pi^2 (c_1^+)^2 - 8\pi^2 (2\chi + 3\tau)(X),$$

where r is the Ricci curvature of g, and $\chi(X)$ and $\tau(X)$ denote the Euler characteristic and the signature of X, respectively.

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When $c_1^+ \neq 0$, the equality in each case is attained iff g is a Kähler metric of negative constant scalar curvature with the Kähler form being a multiple of c_1^+.

Here, a class $\alpha \in H^2(X, \mathbb{Z})$ is called a *monopole class* if it arises as the first Chern class of a Spin^c structure for which the Seiberg–Witten equations admit a solution for every choice of a Riemannian metric on X.

These inequalities lead to a vanishing theorem for the Seiberg–Witten invariant on a 4-manifold with an *F-structure* which was introduced by Cheeger and Gromov [4,5] generalizing an effective torus action.

Definition 1. An F-structure on a smooth manifold is given by data $(U_i, \hat{U}_i, T^{k_i})$ with the following conditions:

(1) $\{U_i\}$ is a locally finite open cover.
(2) $\pi_i : \hat{U}_i \mapsto U_i$ is a finite Galois covering with covering group Γ_i.
(3) A torus T^{k_i} of dimension k_i acts effectively on $\hat{U}_i$ in a Γ_i-equivariant way, i.e., Γ_i also acts on T^{k_i} as an automorphism so that

$$\gamma(gx) = \gamma(g)\gamma(x)$$

for any $\gamma \in \Gamma_i$, $g \in T^{k_i}$, and $x \in \hat{U}_i$.
(4) If $U_i \cap U_j \neq \emptyset$, then there is a common covering of $\pi_i^{-1}(U_i \cap U_j)$ and $\pi_j^{-1}(U_i \cap U_j)$ such that the lifted actions of T^{k_i} and T^{k_j} commute.

An F-structure is called *polarized* if each T^{k_i} action is locally free, and of *positive rank* if every orbit is of positive dimension. They have shown that a smooth compact manifold admits an F-structure of positive rank iff it admits a sequence of Riemannian metrics g_δ such that, as $\delta \to 0$, the injectivity radius converges uniformly to 0 at all points while the sectional curvatures stay uniformly bounded. Paternain and Petean [15] showed that a smooth compact manifold with an F-structure admits a sequence of Riemannian metrics with volume form converging to zero uniformly while the sectional curvatures are bounded below.

Now the first inequality in the above theorem implies that if X admits an F-structure, then a monopole class cannot arise for any Spin^c structure with $c_1^2 > 0$. According to Paternain and Petean [15], every compact Kähler surface of Kodaira dimension 0 or 1 admits an F-structure. Whether this is still true for symplectic 4-manifolds is not known yet, and there are minimal symplectic 4-manifolds X of Kodaira dimension 0 or 1 which admit a polarized F-structure so that they have a sequence of metrics $\{g_i\}$ with

$$\int_X s_{g_i}^2 \mathrm{d}\mu_{g_i} \to 0 = 32\pi^2 c_1^2(X),$$

and

$$\int_X |r_{g_i}|^2 \mathrm{d}\mu_{g_i} \to 0 = 16\pi^2 c_1^2(X) - 8\pi^2(2\chi + 3\tau)(X),$$

but they never admit a Kähler or even a complex structure. Fernández, Gotay and Gray's example [6] is a T^2-bundle over T^2 admitting a free S^1-action and Gompf's example [8] is simply connected. He constructed it by taking a symplectic sum of simply connected elliptic surfaces $E(1)_p$ and $E(1)_q$ along generic fibers of tori, where the gluing map preserves a local S^1-action to give a global polarized F-structure.

Now we delve into the case of 3-manifolds. Similar curvature estimates also hold.

Theorem 1.2. *Let (M, g) be a smooth closed oriented Riemannian 3-manifold with $b_1(M) \geq 1$. Suppose that (M, g) has a solution of the Seiberg–Witten equations for a Spin^c structure with the first Chern class c_1, and $[\omega]$ is a nonzero element in $H^1_{DR}(M)$. Then*

$$\int_M (s_-)^2 \mathrm{d}\mu \geq \frac{16\pi^2 |c_1 \cup [\omega]|^2}{\int_M |\omega|^2 \mathrm{d}\mu}. \tag{1}$$

If the Seiberg–Witten invariant of the Spin^c structure is nonzero, then

$$\int_M |r|^2 \mathrm{d}\mu \geq \frac{8\pi^2 |c_1 \cup [\omega]|^2}{\int_M |\omega|^2 \mathrm{d}\mu}. \tag{2}$$

When $[c_1] \neq 0 \in H^2_{DR}(M)$, the equality in each case holds iff $(M \times S^1, g + \mathrm{d}t^2)$ is a Kähler metric of negative constant scalar curvature with the Kähler form a multiple of

$$*_g\, c_1^h + c_1^h \wedge \mathrm{d}t,$$

and $\omega = \omega^h$ is a multiple of $*_g\, c_1^h$, where $*_g$ is the Hodge star with respect to g, and $(\cdot)^h$ denotes the harmonic part.

When $[c_1] = 0 \in H^2_{DR}(M)$, if we assume that (M, g) has a solution of any perturbed Seiberg–Witten equations for such c_1, then the equality in each case holds iff (M, g) is a flat manifold T^3/Γ such that $(M \times S^1, g + \mathrm{d}t^2)$ is Kähler with the Kähler form a multiple of

$$*_g\, \omega^h + \omega^h \wedge \mathrm{d}t.$$

As in the case of symplectic 4-manifolds, we show an example of a contact 3-manifold which has a sequence of Riemannian metrics whose L^2-norm of curvature converges to the lower bound given by the contact structure but does not admit a Kähler structure on $M \times S^1$.

Theorem 1.3. *Let M be a closed oriented 3-manifold which fibers over the circle with the fiber F being a Riemann surface of Euler characteristic $\chi(F) \leq 0$. Let $\pi : M \to S^1$ be the projection map and $[\omega]$ be a nonzero element of $H^1_{DR}(S^1)$. Then there exists a sequence of metrics $\{g_i\}$ on M such that*

$$\left(\int_M s_{g_i}^2 \mathrm{d}\mu_{g_i}\right)\left(\int_M |\pi^*\omega|_{g_i}^2 \mathrm{d}\mu_{g_i}\right) \to 16\pi^2 |c_1 \cup [\pi^*\omega]|^2,$$

and

$$\left(\int_M |r_{g_i}|^2 \mathrm{d}\mu_{g_i}\right)\left(\int_M |\pi^*\omega|_{g_i}^2 \mathrm{d}\mu_{g_i}\right) \to 8\pi^2 |c_1 \cup [\pi^*\omega]|^2.$$

Here c_1 is a monopole class $\chi(F)[F]$ given by the contact structure near the 2-planes of the fibers.

But for a certain monodromy $M \times S^1$ never admits a Kähler structure.

The L^2-estimate of the scalar curvature on a 3-manifold can also give conditions to be a monopole class on F-structured 3-manifolds, also known as graph manifolds. Indeed a collapsing sequence of metrics from the Cheeger–Gromov theory [4] gives a restriction to be a monopole class. First, it is easy to see that Theorem 1.2 says that a monopole class should have intersection number zero with any $[\omega]$ such that $[\omega]$ restricts to zero on each circle given by the orbits of the F-structure. This method also applies to a connected sum of graph manifolds and any other 3-manifolds, and the 3-manifolds obtained by gluing hyperbolic 3-manifolds and graph manifolds along tori. But noting that the Poincaré-dual of such $[\omega]$ can be represented by an embedded torus, these facts also follow from the well-known adjunction inequality. An interesting application is the following one.

Theorem 1.4. *Let M be a closed oriented 3-manifold which fibers over the circle with a periodic monodromy, and N be a closed oriented 3-manifold. Then the rational part of a monopole class of $M\#N$ is of the form $m[F]$ for an integer m satisfying $|m| \leq |\chi(F)|$, where $\chi(F)$ is the Euler characteristic of the fiber F.*

Theorem 1.2 has a relation to the Yamabe problem too. The *Yamabe constant* $Y(M, [g])$ of a conformal class $[g] \equiv \{e^f g | f \in C^\infty(M)\}$ is defined as

$$\inf_{\tilde{g} \in [g]} \frac{\int_M s_{\tilde{g}}\, \mathrm{d}\mu_{\tilde{g}}}{(\mathrm{Vol}_{\tilde{g}})^{\frac{n-2}{n}}},$$

where $n = \dim M$ and $\mathrm{Vol}_{\tilde{g}} = \int_M \mathrm{d}\mu_{\tilde{g}}$, and the *Yamabe invariant* $Y(M)$ of M is defined as

$$\sup_{[g]} Y(M, [g]),$$

where the supremum is taken for all the conformal classes on M.

Theorem 1.5. *Let (M, g) be a closed oriented Riemannian 3-manifold. Suppose c_1 is a non-torsion monopole class. Then*

$$Y(M,[g]) \le -\frac{4\pi|c_1 \cup [\omega]|}{(\int_M |\omega|_g^3 \, \mathrm{d}\mu_g)^{\frac{1}{3}}}$$

for any nonzero $[\omega] \in H^1_{DR}(M)$*, with the equality iff* $(M \times S^1, g + \mathrm{d}t^2)$ *is a Kähler metric of negative constant scalar curvature with the Kähler form a multiple of* $*_g\,\omega + \omega \wedge \mathrm{d}t$.

A maximizing sequence of the Yamabe constants is supposed to reveal the structure of a 3-manifold [1]. For a closed 3-manifold M with $Y(M) \le 0$,

$$Y(M) = -\inf_{\tilde{g}\in\mathbb{M}_1} \left(\int_M (s_-)^2_{\tilde{g}} \mathrm{d}\mu_{\tilde{g}}\right)^{\frac{1}{2}},$$

where $\mathbb{M}_1$ is the space of smooth unit-volume Riemannian metrics on M. With this in mind we have

Theorem 1.6. *Let M be a closed oriented 3-manifold. Suppose there exists a non-torsion monopole class. Let (g_i) be any sequence of unit-volume Riemannian metrics on M such that $\int_M (s_-)^2_{g_i} \mathrm{d}\mu_{g_i} \to 0$. Then the injectivity radius converges to zero as $i \to \infty$.*

2. A brief review of Seiberg–Witten theory on the 3-manifold

Let (M, g) be a closed oriented Riemannian 3-manifold. A Spin^c structure ξ on M is a Spin$^c(3)$-lift of the orthonormal frame bundle. We denote the associated $U(2)$-bundle by W and its determinant line bundle by L. We choose a unique action of the complex Clifford algebra bundle $Cl(T^*Y) \otimes \mathbb{C}$ on W such that the volume form acts as either Id or $-$Id. The Levi-Civita connection on TM and a $U(1)$-connection A on L induce a Spin^c connection on W and the associated Dirac operator $D_A : \Gamma(W) \to \Gamma(W)$ is given by

$$D_A = \sum_i e_i \cdot \nabla^A_{e_i},$$

where $\{e_i\}$ is an orthonormal frame of TM, $\cdot$ denotes the Clifford action, and ∇^A is the covariant derivative of the Spin^c connection.

For a section Φ of W, a perturbed Seiberg–Witten equation of (A, Φ) is given by

$$(*) \begin{cases} D_A\Phi = 0 \\ F_A + i * \omega = \Phi \otimes \Phi^* - \dfrac{|\Phi|^2}{2}\mathrm{Id}, \end{cases}$$

where $F_A = dA$ is the curvature of A, and a real-valued coclosed 1-form ω is a perturbation term. In the second equation the identification of the two sides comes from the isomorphism between $\wedge^2(M) \otimes i\mathbb{R}$ and $su(W) \otimes i\mathbb{R}$ given by the Clifford action.

Since the gauge transformation group $\mathcal{G} \equiv \{e^{if} \mid f \in C^\infty(M)\}$ acts on the solution space of $(*)$, the Seiberg–Witten moduli space $\mathcal{M}$ is defined as the space of solutions of $(*)$ modulo $\mathcal{G}$. In the topology induced from an appropriate Sobolev space, $\mathcal{M}$ is compact, and moreover it is a smooth 0-dimensional manifold for a generic choice of ω if $b_1(M) > 0$.

Now a numeric topological invariant of M is obtained by counting $\mathcal{M}$ in an appropriate way. For instance, we give an orientation on $\mathcal{M}$ from a fixed orientation of $H^2(M, \mathbb{R})$. Then the Seiberg–Witten invariant $SW(M, \xi)$ is independent of the choice of g and ω, if $b_1(M) > 1$. In the case of $b_1(M) = 1$, it depends only on the component of $H^2(M, \mathbb{R}) - \{c_1(L)_\mathbb{R}\}$ in which $[-\frac{1}{2\pi} * \omega]$ lies. Even when $b_1(M) = 0$, by adding a counter-term given by the spectral invariants of Atiyah, Patodi and Singer, an invariant independent of g and ω can be defined. For more details, readers are referred to [14].

3. A proof of Theorem 1.2

Put the product metric $g + \mathrm{d}t^2$ on $M \times S^1$, where $t \in [0, 1]$ is a global coordinate of S^1. Note that any solution of the Seiberg–Witten equations of (M, g) is a t-invariant solution of the corresponding Seiberg–Witten equations on $(M \times S^1, g + \mathrm{d}t^2)$, and $SW(M, \xi) = SW(M \times S^1, \pi^*\xi)$ for any Spin^c structure ξ.

Now let us apply those estimates in Theorem 1.1; we get

$$\int_M (s_-)^2_g \, \mathrm{d}\mu_g = \int_{M\times S^1} (s_-)^2_{g+\mathrm{d}t^2}\mathrm{d}\mu_{g+\mathrm{d}t^2} \geq 32\pi^2((\pi^* c_1)^+)^2 \tag{3}$$

$$= 32\pi^2 \left(\frac{\pi^* c_1^h + \pi^*(*_g c_1^h)\wedge \mathrm{d}t}{2}\right)^2$$

$$= 16\pi^2 \langle \pi^* c_1^h \wedge \pi^*(*_g c_1^h) \wedge \mathrm{d}t, [M\times S^1]\rangle$$

$$= 16\pi^2 \langle c_1^h \wedge *_g c_1^h, [M]\rangle$$

$$\geq \frac{16\pi^2 |c_1 \cup [\omega]|^2}{\int_M |\omega|^2_g \, \mathrm{d}\mu_g}, \tag{4}$$

where the last inequality is a combination of the Hölder inequality and the fact that $\int_M |\omega|^2_g \, \mathrm{d}\mu_g \geq \int_M |\omega^h|^2_g \, \mathrm{d}\mu_g$. In the Ricci curvature case, from the nonvanishing of the Seiberg–Witten invariant of $M \times S^1$ for $\pi^* c_1$, $\pi^* c_1$ is a monopole class of $M\times S^1$. Applying Theorem 1.1 with the fact that $\chi(M\times S^1) = \tau(M\times S^1) = 0$, we get

$$\int_M |r_g|^2 \, \mathrm{d}\mu_g = \int_{M\times S^1} |r_{g+\mathrm{d}t^2}|^2 \, \mathrm{d}\mu_{g+\mathrm{d}t^2} \geq 16\pi^2((\pi^* c_1)^+)^2 \tag{5}$$

$$\geq \frac{8\pi^2 |c_1 \cup [\omega]|^2}{\int_M |\omega|^2_g \, \mathrm{d}\mu_g}. \tag{6}$$

To decide the equality case, first let us consider the case when $[c_1] \neq 0 \in H^2_{DR}(M)$. Note from [13] that the equality of the 4-dimensional case in (5) holds iff the equality in (3) holds. The equality (3) or (5) holds iff $g + \mathrm{d}t^2$ is a Kähler metric of negative constant scalar curvature with the Kähler form a multiple of

$$(\pi^* c_1)^+ = \frac{\pi^*(*_g c_1^h) + \pi^* c_1^h \wedge \mathrm{d}t}{2}.$$

The equalities (4) and (6) hold iff

$$\omega = \omega^h = *_g c_1^h.$$

Now let us suppose $[c_1] = 0 \in H^2_{DR}(M)$. First, the equalities (4) and (6) are automatic. Obviously the equality in (5) holds iff $g + \mathrm{d}t^2$ is Ricci-flat, which is again iff g is flat.

If the equality in (3) holds, $(s_-)_g \equiv 0$. We claim that $s_g \equiv 0$. Consider

Lemma 3.1 (*LeBrun [12]*). *Let (X, g) be a smooth closed oriented Riemannian 4-manifold with $b_2^+(X) \geq 1$, and Ω be a nonzero self-dual harmonic 2-form. Suppose that ξ is a* Spin^c *structure with first Chern class c_1 and its Seiberg–Witten equations perturbed by $ir\Omega$ have a solution for any $r \in \mathbb{R}$. Then*

$$\int_X s\frac{|\Omega|}{\sqrt{2}} \, \mathrm{d}\mu \leq 4\pi c_1 \cup [\Omega].$$

The equality holds iff it is Kähler with the Kähler form a positive multiple of Ω.

Applying this lemma to $(M\times S^1, g + \mathrm{d}t^2)$ with $\Omega = \pi^*(*_g \tilde{\omega}^h) + \pi^* \tilde{\omega}^h \wedge \mathrm{d}t$ for any nonzero $[\tilde{\omega}] \in H^1_{DR}(M)$, the equality in (3) holds iff $g+\mathrm{d}t^2$ is a scalar-flat Kähler metric with the Kähler form a multiple of $\pi^*(*_g \tilde{\omega}^h) + \pi^*\tilde{\omega}^h \wedge \mathrm{d}t$. On scalar-flat Kähler surfaces, the self-dual Weyl curvature W_+ is zero (see [2]), and hence the 4-dimensional Chern–Gauß–Bonnet theorem

$$\frac{1}{8\pi^2}\int_{M\times S^1} |r_{g+\mathrm{d}t^2}|^2 \mathrm{d}\mu_{g+\mathrm{d}t^2} = \frac{1}{\pi^2}\int_{M\times S^1}\left(\frac{1}{24}(s_{g+\mathrm{d}t^2})^2 + \frac{1}{2}|W_+|^2_{g+\mathrm{d}t^2}\right)\mathrm{d}\mu_{g+\mathrm{d}t^2} - (2\chi + 3\tau)(M\times S^1)$$

$$= 0$$

implies that $g + \mathrm{d}t^2$ is Ricci-flat, i.e., g is flat. By the Bieberbach theorem [18], every closed flat 3-manifold is a finite quotient of T^3. This completes the proof.

Remark. In the above $T^3/\Gamma \times S^1$ is actually T^4 or a hyperelliptic surface. Since the first Chern class of the canonical line bundle is a torsion, the Kodaira dimension of X is zero. By the Enriques–Kodaira classification of Kähler surfaces, there are four types: Enriques surfaces, K3 surfaces, complex tori and hyperelliptic surfaces. Since the first two types have $b_1 = 0$, they are excluded. All hyperelliptic surfaces have $b_2^+ = 1$ so that our T^3/Γ should have $b_1 = 1$ except for T^3. □

Remark. When $*_g\, \omega + \omega \wedge \mathrm{d}t$ is parallel, ω is g-parallel because $\mathrm{d}t$ is parallel too. If ω is parallel, ω and $*_g\, \omega$ give a parallel splitting of TM so that g is locally a product metric. Therefore a Riemannian 3-manifold (M, g) such that $(M \times S^1, g + \mathrm{d}t^2, *_g\, \omega + \omega \wedge \mathrm{d}t)$ for $\omega \in \wedge^1(M)$ is a Kähler metric of constant scalar curvature is a quotient of $S^2 \times \mathbb{R}$, or $\mathbb{R}^3$, or $\mathbb{H}^2 \times \mathbb{R}$ with the obvious product metric. □

It seems very plausible to conjecture the following:

Conjecture 3.2. *The same Ricci curvature estimate also holds for monopole classes.*

4. A proof of Theorem 1.3

Let $t \in [0, 1]$ be a global coordinate of the base S^1 with a metric $\mathrm{d}t^2$. Put a smooth metric g on M such that π is a Riemannian submersion. Write $g = h_t + \pi^*\mathrm{d}t^2$, where h_t is any smooth S^1-parameter family of Riemannian metrics on the fiber F. Since $\chi(F) \le 0$, by the Poincaré uniformization theorem, each fiber with the metric h_t admits a unique smooth conformal change to a metric of constant curvature and volume 1. Let e^{φ_t} be such a conformal factor for h_t.

Lemma 4.1. *As t varies, e^{φ_t} defines a smooth function on M.*

Proof. Since the smoothness is a local property, we will consider $I \times F \subset M$ for an interval I. Suppose $\chi(F) < 0$. Then the equation that φ_t satisfies is

$$P_t(\varphi_t) \equiv \Delta_t \varphi_t - 4\pi\chi(F)e^{\varphi_t} + s_t = 0, \tag{7}$$

where Δ_t and s_t are the Hodge Laplacian and the scalar curvature of the metric h_t respectively. Let $C^{k,\alpha}(F)$ for $k \in \mathbb{N}$ and $\alpha \in (0, 1]$ be the Banach space of real-valued continuous functions on F with bounded $C^{k,\alpha}$-norm. The linearization $\bar{P}_t$ of the smooth map $P_t : C^{k+2,\alpha}(F) \to C^{k,\alpha}(F)$ at φ_t is

$$\Delta_t - 4\pi\chi(F)e^{\varphi_t}(\cdot)$$

which is a self-adjoint elliptic operator, and its index is equal to 0. Let $\psi \in C^{k,\alpha}(F)$ be an element of the kernel of $\bar{P}_t$. Since $\chi(F) < 0$, $\psi = 0$, implying that $\bar{P}_t$ is an isomorphism. Considering a smooth map P

$$P : I \times C^{k+2,\alpha}(F) \to C^{k,\alpha}(F),$$

where $P(t, \psi) = P_t(\psi)$, the solution space $P^{-1}(0)$ is a smooth 1-dimensional submanifold of $I \times C^{k+2,\alpha}(F)$ by the Banach-space inverse function theorem.

When $\chi(F) = 0$, the equations that φ_t satisfy are

$$P_t(\varphi_t) = \Delta_t \varphi_t + s_t = 0, \quad \text{and} \quad Q_t(\varphi_t) \equiv \int_F e^{\varphi_t}\,\mathrm{d}\mu_{h_t} - 1 = 0.$$

The kernel and cokernel of the linearization $\bar{P}_t$ of P_t are $\mathbb{R} \cdot 1$. Consider a smooth map

$$(\Pi \circ P_t, Q_t) : C^{k+2,\alpha}(F) \to C^{k,\alpha}(F)/\mathbb{R} \oplus \mathbb{R},$$

where $\Pi : C^{k,\alpha}(F) \to C^{k,\alpha}(F)/\mathbb{R}$ is the projection map. Then the linearization of $(\Pi \circ P_t, Q_t)$ is an isomorphism. In the same way as above, considering a smooth map

$$P : I \times C^{k+2,\alpha}(F) \to C^{k,\alpha}(F)/\mathbb{R} \oplus \mathbb{R},$$

where $P(t, \psi) = (\Pi \circ P_t, Q_t)$, the solution space given by $P^{-1}(0)$ is a smooth 1-dimensional submanifold.

Now the proof proceeds regardless of $\chi(F)$. Define a map

$$\Phi_1 : I \times F \to C^{k+2,\alpha}(F) \times F$$

by $\Phi_1(t, x) = (\varphi_t, x)$ for $(t, x) \in I \times F$. By the above result, Φ_1 is a smooth map. Also consider a map

$$\Phi_2 : C^{k+2,\alpha}(F) \times F \to \mathbb{R},$$

where $\Phi_2(\psi, x) = \psi(x)$ for $(\psi, x) \in C^{k+2,\alpha}(F) \times F$. One can easily check that Φ_2 is C^{k+2}-differentiable. Therefore the composition $\Phi_2 \circ \Phi_1 : I \times F \to \mathbb{R}$, which is our anticipated function, is a C^{k+2}-function. Since k is arbitrary, it has to be a smooth function. □

Define a new smooth metric $\tilde{g}_l$ on M for a constant $l > 0$ by

$$\tilde{g}_l = e^{\varphi_t} h_t + l^2 \pi^* \mathrm{d}t^2.$$

As l tends to ∞, the sectional curvature of the plane spanned by a vertical vector and a horizontal vector converges to 0, and the sectional curvature of the plane spanned by vertical vectors converges to the sectional curvature of $(F, e^{\varphi_t} h_t)$ which is $2\pi\chi(F)$. Therefore for any $\varepsilon > 0$, there exists $L > 0$ such that $|s_{\tilde{g}_l} - 4\pi\chi(F)| < \varepsilon$ on M for $l \geq L$, where $4\pi\chi(F)$ is the scalar curvature of $e^{\varphi_t} h_t$.

We will denote the volume element $\mathrm{d}\mu_{e^{\varphi_t} h_t}$ on F simply by $\mathrm{d}\mu_t$. Using the fact that s_t is constant on F for each t and $\int_F \mathrm{d}\mu_t = 1$, for $l \geq L$

$$\begin{aligned}
\int_M s_{\tilde{g}_l}^2 \mathrm{d}\mu_{\tilde{g}_l} &\leq \int_M (|4\pi\chi(F)| + \varepsilon)^2 \mathrm{d}\mu_{\tilde{g}_l} \\
&= (|4\pi\chi(F)| + \varepsilon)^2 \int_{S^1} \int_F l \, \mathrm{d}\mu_t \mathrm{d}t \\
&= (|4\pi\chi(F)| + \varepsilon)^2 l,
\end{aligned}$$

and

$$\int_M |\pi^* \mathrm{d}t|_{\tilde{g}_l}^2 \mathrm{d}\mu_{\tilde{g}_l} = \int_{S^1} \int_F \frac{1}{l^2} \, l \, \mathrm{d}\mu_t \mathrm{d}t = \frac{1}{l}.$$

So

$$\left(\int_M s_{\tilde{g}_l}^2 \mathrm{d}\mu_{\tilde{g}_l} \right) \left(\int_M |\pi^* \mathrm{d}t|_{\tilde{g}_l}^2 \mathrm{d}\mu_{\tilde{g}_l} \right) \leq (|4\pi\chi(F)| + \varepsilon)^2.$$

Likewise we get

$$\left(\int_M s_{\tilde{g}_l}^2 \mathrm{d}\mu_{\tilde{g}_l} \right) \left(\int_M |\pi^* \mathrm{d}t|_{\tilde{g}_l}^2 \mathrm{d}\mu_{\tilde{g}_l} \right) \geq (|4\pi\chi(F)| - \varepsilon)^2.$$

Therefore, as $l \to \infty$,

$$\left(\int_M s_{\tilde{g}_l}^2 \mathrm{d}\mu_{\tilde{g}_l} \right) \left(\int_M |\pi^* \mathrm{d}t|_{\tilde{g}_l}^2 \mathrm{d}\mu_{\tilde{g}_l} \right) \to |4\pi\chi(F)|^2.$$

For a proof that $\chi(F)[F]$ is a monopole class, one may refer to [10]. And

$$|c_1 \cup [\pi^* \mathrm{d}t]| = |\chi(F)|,$$

because $[F]$ is the Poincaré-dual of $[\pi^* \mathrm{d}t]$. The Ricci curvature estimate is also obtained in the same way.

For an example without a Kähler structure, one can easily construct such M with $b_1(M)$ even so that $M \times S^1$ never admits a Kähler structure, although it admits a symplectic structure compatible with the contact structure of M. Thurston's example [17] is one of this kind.

Remark. One can also show that $\|\pi^* \mathrm{d}t - (\pi^* \mathrm{d}t)^h\|_{L^2} \to 0$ as $l \to \infty$. □

Remark. One may expect to find M such that $M \times S^1$ cannot admit even a complex structure. □

5. A proof of Theorem 1.4

Let f be the monodromy diffeomorphism and $\pi : M \to S^1$ be the projection map. Also let $t \in [0, 1]$ be a coordinate of the base S^1. The nontrivial first de Rham cohomologies of M and hence $M\#N$ come from

$$\pi^* dt, \quad \text{and} \quad \{\sigma \in H^1_{DR}(F) | f^*\sigma \text{ is cohomologous to } \sigma\}.$$

Indeed the latter ones can be explicitly expressed as

$$\frac{1}{|G|} \sum_{f_i \in G} f_i^* \sigma$$

for each such σ, where G is the finite group generated by f.

First, since $[F] \in H_2(M\#N, \mathbb{Z})$ is the Poincaré-dual to $\pi^* dt$, from the adjunction inequality, it easily follows that the intersection number of a monopole class and $\pi^* dt$ is an integer between $\chi(F)$ and $-\chi(F)$. (In fact one can also derive this by using our inequality of Theorem 1.2.) Since $[F]$ is not a positive integral multiple of any other element of $H_2(M, \mathbb{Z})$, m must be an integer between $\chi(F)$ and $-\chi(F)$.

To show that the coefficients for other cohomology classes all vanish, we need to use a collapsing sequence of metrics on M. Take any Riemannian metric on F and let g_F be the metric obtained by taking the average of the initial metric by the G-action, so that f is an isometry of g_F. Put the metric $\varepsilon^2 dt^2$ on the base S^1. Now we can define a smooth metric g_ε on M as the Riemannian submersion onto $(S^1, \varepsilon^2 dt^2)$ with totally geodesic fibers (F, g_F). Since the O'Neill tensor is zero, g_ε is a locally product metric. Let $C_1 \equiv \int_F (s_-)^2_{g_F} d\mu_{g_F} > 0$ so that

$$\int_M (s_-)^2_{g_\varepsilon} d\mu_{g_\varepsilon} = C_1 \varepsilon.$$

On N we put any metric h so that

$$\int_M (s_-)^2_h \, d\mu_h \le C_2$$

for a constant $C_2 > 0$.

For $[\sigma] \in H^1_{DR}(F)$ we can take its representative σ such that $\omega \equiv \frac{1}{|G|} \sum_{f_i \in G} f_i^* \sigma$ is identically zero on an open ball $B \subset F$. Let $D \equiv \int_F |\omega|^2_{g_F} d\mu_{g_F} > 0$. Denoting the 1-form on M coming from ω also by ω, ω is zero on an open set $B \times (-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}) \subset M$.

Now take a connected sum $M\#N$ by performing the Gromov–Lawson method of surgery [9,16] on $B \times (-\frac{\varepsilon}{3}, \frac{\varepsilon}{3})$ such that a resulting metric $g_\varepsilon \# h$ satisfies

$$\int_{M\#N} (s_-)^2_{g_\varepsilon \# h} \, d\mu_{g_\varepsilon \# h} \le (C_1 \varepsilon + C_2 + 1),$$

and

$$\int_{M\#N} |\omega|^2_{g_\varepsilon \# h} \, d\mu_{g_\varepsilon \# h} = D\varepsilon.$$

Then by Theorem 1.2

$$|4\pi c_1 \cup [\omega]| \le (C_1 \varepsilon + C_2 + 1) D\varepsilon.$$

As $\varepsilon \to 0$, we get the desired conclusion.

Remark. When $N = S^3$, from the above construction one can see that $M \times S^1$ has a Kähler structure of elliptic type with the canonical line bundle equal to $-\chi(F)[F]$. In such a case the result also follows from the computation of the monopole classes on Kähler surfaces [7].

6. A proof of Theorem 1.5

It is well known that on any closed n-manifold the Yamabe constant is always achieved by a metric of constant scalar curvature, which we call a Yamabe minimizer. Using the technique of Besson, Courtois, and Gallot [3], the

Yamabe constant can be also written as:

Lemma 6.1. *Let* $[g]$ *be a conformal class on a closed n-manifold* M *with* $n \geq 3$. *Then for* $r \in [\frac{n}{2}, \infty]$,

$$|Y(M,[g])| = \inf_{\tilde{g}\in[g]} \left(\int_M |s_{\tilde{g}}|^r \mathrm{d}\mu_{\tilde{g}}\right)^{\frac{1}{r}} (\mathrm{Vol}_{\tilde{g}})^{\frac{2}{n}-\frac{1}{r}}$$

where the infimum is realized only by a Yamabe minimizer.

Proof. When $Y(M,[g]) \geq 0$, the proof is easy. Simply by the Hölder inequality,

$$Y(M,[g]) = \inf_{\tilde{g}\in[g]} \frac{\int_M s_{\tilde{g}}\, \mathrm{d}\mu_{\tilde{g}}}{(\mathrm{Vol}_{\tilde{g}})^{\frac{n-2}{n}}} \leq \inf_{\tilde{g}\in[g]} \frac{(\int_M |s_{\tilde{g}}|^r \mathrm{d}\mu_{\tilde{g}})^{\frac{1}{r}} (\mathrm{Vol}_{\tilde{g}})^{1-\frac{1}{r}}}{(\mathrm{Vol}_{\tilde{g}})^{\frac{n-2}{n}}}$$

with the equality iff $\tilde{g}$ is a Yamabe minimizer.

Now suppose $Y(M,[g]) \leq 0$ and g is a Yamabe minimizer. Write $\tilde{g} = u^{p-2}g$, where $p = \frac{2n}{n-2}$ and $u > 0$. Recall that $s_{\tilde{g}}u^{p-1} = s_g u + 4\frac{n-1}{n-2}\Delta_g u$. Therefore

$$\begin{aligned}
\left(\int_M |s_{\tilde{g}}|^r \mathrm{d}\mu_{\tilde{g}}\right)^{\frac{1}{r}} (\mathrm{Vol}_{\tilde{g}})^{\frac{2}{n}-\frac{1}{r}} &= \left(\int_M |s_{\tilde{g}}|^r u^{\frac{n(p-2)}{2}} \mathrm{d}\mu_g\right)^{\frac{1}{r}} \left(\int_M u^{\frac{n(p-2)}{2}} \mathrm{d}\mu_g\right)^{\frac{2}{n}-\frac{1}{r}} \\
&\geq \frac{\int_M -s_{\tilde{g}} u^{p-2} \mathrm{d}\mu_g}{\left(\int_M \mathrm{d}\mu_g\right)^{\frac{n-2}{n}}} \\
&= \frac{\int_M -(s_g + 4\frac{n-1}{n-2}\frac{1}{u} d^*\mathrm{d}u)\mathrm{d}\mu_g}{(\mathrm{Vol}_g)^{\frac{n-2}{n}}} \\
&= \frac{\int_M \left(-s_g + 4\frac{n-1}{n-2}\frac{|\mathrm{d}u|^2}{u^2}\right)\mathrm{d}\mu_g}{(\mathrm{Vol}_g)^{\frac{n-2}{n}}} \\
&\geq \frac{\int_M -s_g \mathrm{d}\mu_g}{(\mathrm{Vol}_g)^{\frac{n-2}{n}}} = -Y(M,[g]),
\end{aligned}$$

where the first inequality is an application of the Hölder inequality, and the equality holds iff u is a positive constant. □

Since the monopole class c_1 is non-torsion, from the estimate of Theorem 1.2 M cannot admit a positive scalar curvature metric, i.e., $Y(M) \leq 0$. The proof of the theorem immediately follows from Theorem 1.2 and the above lemma as

$$\begin{aligned}
Y(M,[g]) &= -\inf_{\tilde{g}\in[g]} \left(\int_M |s_{\tilde{g}}|^2 \mathrm{d}\mu_{\tilde{g}}\right)^{\frac{1}{2}} (\mathrm{Vol}_{\tilde{g}})^{\frac{1}{6}} \\
&\leq -\inf_{\tilde{g}\in[g]} \frac{4\pi|c_1 \cup [\omega]|}{(\int_M |\omega|^2_{\tilde{g}}\, \mathrm{d}\mu_{\tilde{g}})^{\frac{1}{2}}} \left(\int_M \mathrm{d}\mu_{\tilde{g}}\right)^{\frac{1}{6}} \\
&\leq -\inf_{\tilde{g}\in[g]} \frac{4\pi|c_1 \cup [\omega]|}{(\int_M |\omega|^3_{\tilde{g}}\, \mathrm{d}\mu_{\tilde{g}})^{\frac{1}{3}}} = -\frac{4\pi|c_1 \cup [\omega]|}{(\int_M |\omega|^3_g\, \mathrm{d}\mu_g)^{\frac{1}{3}}},
\end{aligned}$$

where the last equality comes from the fact that $|\omega|^3_g \mathrm{d}\mu_g$ is conformally invariant. This completes the proof.

Remark. If (M,g) is hyperbolic and has a solution for the Seiberg–Witten equations, we actually have a better estimate. From LeBrun's estimate [13],

$$(\mathrm{Vol}_{g+dt^2})^{\frac{1}{3}} \left(\int_{M\times S^1} \left|\frac{2}{3}s_{g+dt^2} - 2\sqrt{\frac{2}{3}}|W_+|_{g+dt^2}\right|^3 \mathrm{d}\mu_{g+dt^2}\right)^{\frac{2}{3}} \geq 32\pi((\pi^* c_1)^+)^2.$$

One can easily show that $W_+ = 0$, and hence

$$\frac{4}{9}s_g^2 \cdot \mathrm{Vol}_g \geq \frac{16\pi^2 |c_1 \cup [\omega]|^2}{\int_M |\omega|^2 \mathrm{d}\mu}.$$

7. A proof of Theorem 1.6

We can take a smooth closed oriented surface Σ embedded in M representing the dual of the monopole class.

Suppose the contrary, i.e., there exists a constant $\delta > 0$ such that the injectivity radius of g_i is greater than δ for all i. Let E be a normal bundle of Σ in M and put the Euclidean metric on each fiber. Letting the δ-neighborhood of the zero section in E be $N(\delta)$, each g_i-exponential map $N(\delta)$ diffeomorphically onto a neighborhood of Σ in M.

Note that E is a trivial bundle, since M and Σ are oriented. So we let $E = \Sigma \times \mathbb{R}$, and take a representative for the Thom class, $\omega = \rho(r)dr$, where a smooth nonnegative function $\rho(r)$ defined on $\mathbb{R}$ is supported on $[-\delta, \delta]$ and has total mass 1.

We claim that

$$|\omega|_{g_i} = \rho(r).$$

First, $|\frac{\partial}{\partial r}|_{g_i} = 1$, and it follows from the Gauss lemma that

$$\omega\left(\frac{\partial}{\partial r}\right) = \rho(r).$$

Let v be any tangent vector orthogonal to $\frac{\partial}{\partial r}$. We have to show that ω is zero on v. Letting $\pi : E \to \Sigma$ be the projection map, $\pi_* v$ is orthogonal to $\frac{\partial}{\partial r}$. Applying the first variation formula of the arc length to the 1-parameter family of our geodesics emanating from Σ given by the variation vector $\pi_* v$,

$$\mathrm{d}r(v) = \left\langle v, \frac{\partial}{\partial r}\right\rangle_{g_i} - \left\langle \pi_* v, \frac{\partial}{\partial r}\right\rangle_{g_i} = 0.$$

This proves our claim, and hence we have for any i

$$\int_M |\omega|_{g_i}^2 \mathrm{d}\mu_{g_i} \leq \int_M C^2 \mathrm{d}\mu_{g_i} = C^2,$$

for a constant C greater than $\rho(r)$ for any r.

But the assumption that $\int_M (s_-)_{g_i}^2 \mathrm{d}\mu_{g_i} \to 0$ combined with the Theorem 1.2 gives

$$\int_M |\omega|_{g_i}^2 \mathrm{d}\mu_{g_i} \geq \frac{16\pi^2 |c_1 \cup [\omega]|^2}{\int_M (s_-)_{g_i}^2 \mathrm{d}\mu_{g_i}} \to \infty$$

as $i \to \infty$, because c_1 which is the Poincaré-dual of ω is non-torsion. This contradiction completes the proof.

Example. Let M be $\Sigma \times S^1$ where Σ is a compact Riemann surface of negative Euler characteristic $\chi(\Sigma)$. Let ξ be the Spin^c structure determined by the complex structure of Σ with $c_1(\xi) = \chi(\Sigma)[\Sigma]$. Then $SW(M, \xi) = \pm 1$. Since $Y(M) = 0$, $\inf_g \int_M (s_-)_g^2 \, \mathrm{d}\mu_g \, (\mathrm{Vol}_g)^{\frac{1}{3}} = 0$, and any minimizing sequence of unit-volume metrics should develop a collapsing. ◇

Remark. One can also consider the minimizing sequence for the functional $\int_M s^2 \mathrm{d}\mu$ and ask further whether the collapsing appears all over M. □

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